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PUSHING DOWN LOEB-MEASURES

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Abstract

Given an internal, finitely additive measure μ on the nonstandard version *X of a set X , we give a method for pushing the Loeb-measure $L(\mu)$ down to a measure on X , using Boolean σ -homomorphisms. The results are applied to problems concerning measure extensions and weak compactness.

Key words and phrases: Nonstandard analysis, Loeb-measures, σ -homomorphisms, measure extensions, weak convergence.

1. Introduction

In a variety of measure constructions the Loeb-measure of nonstandard analysis ([11]) has proved to be a useful tool on both the intuitive and the technical level. A typical application may look something like this: Consider a net $\{\mu_\alpha\}_{\alpha \in I}$ and a notion of convergence; - a limit measure can then often be obtained from the Loeb-measure $L(\tilde{\mu}_\beta)$ of an infinite element in $^*(\{\mu_\alpha\}_{\alpha \in I})$. However, there is a difficulty to surpass; $L(\tilde{\mu}_\beta)$ is usually defined on the wrong space - e.g. on *X instead of X - and a method must be found for pushing the measure down to the appropriate space. When dealing with Borel-measures the standard part map of the defining topology has been used for this purpose in a number of applications (see e.g. Anderson [1], [2], Anderson and Rashid [3], Loeb [12], Helms and Loeb [4], Henson [6], and Lindstrøm [9]), but to my knowledge no method has yet been suggested for more general situations. The aim of this paper is to describe one such method and illustrating its use by giving applications to measure extension and weak convergence, - a comparison with measures constructed by the standard part method is also given.

We shall work with polysaturated models for nonstandard analysis (see Stroyan and Luxemburg [16]), and we assume that all standard sets we come across are contained in our superstructure. For an introduction to nonstandard measure and probability theory, see Loeb [13].

Many of the ideas and methods used in this paper are closely related to those of Topsøe [17], and I am grateful to Professor Erik Alfsen for showing me Topsøe's work.

2. Internal representations

The main idea is to establish a correspondence between the σ -algebra on the nonstandard space and the σ -algebra on the standard space; - the following theorem is the basic tool for constructing these correspondences. If K is a family of sets, let $\sigma(K)$ denote the σ -algebra generated by K .

Theorem 1. Let K be a family of subsets of a set Y , let X be an internal set, and for each $K \in K$, let K' be an internal subset of X . Let $K' = \{K' : K \in K\}$. Assume that for all finite subsets K_1, \dots, K_n of K and all functions $\epsilon: \{1, \dots, n\} \rightarrow \{-1, 1\}$

$$\bigcap_{i \in \mathbb{N}} \epsilon(i) K'_i = \emptyset \Rightarrow \bigcap_{i \in \mathbb{N}} \epsilon(i) K_i = \emptyset.$$

Then the mapping $\theta: K' \rightarrow K$ defined by $\theta(K') = K$ has a (unique) extension to a σ -homomorphism $\theta: \sigma(K') \rightarrow \sigma(K)$.

Proof: By Theorem 34.1 in Sikorski [15], it is enough to prove that for all countable subsets $\{K_n\}_{n \in \mathbb{N}}$ of K and all $\epsilon: \mathbb{N} \rightarrow \{-1, 1\}$, that $\bigcap_{i \in \mathbb{N}} \epsilon(i) K'_i = \emptyset$ implies $\bigcap_{i \in \mathbb{N}} \epsilon(i) K_i = \emptyset$. But since all $\epsilon(i) K_i$ are internal it follows by polysaturation that $\bigcap_{i \in \mathbb{N}} \epsilon(i) K'_i = \emptyset$ implies $\bigcap_{i \leq m} \epsilon(i) K'_i = \emptyset$ for some $m \in \mathbb{N}$. By assumption $\bigcap_{i \leq m} \epsilon(i) K_i = \emptyset$, and the theorem follows.

If K' and θ satisfies the conditions of Theorem 1, they are called an internal representation of K .

The result above is a trivial extension of Theorem 3 of [7], where we assumed that $K' = {}^*K$; we shall see in the next section that it is useful to allow other choices of K' .

If $A \in \sigma(K)$, let $A^\theta = \{A' \in \sigma(K') : \theta(A') = A\}$. Let μ be an

internal, finitely additive measure on X , such that all $A' \in \sigma(K')$ are $L(\mu)$ -measurable; we want to push $L(\mu)$ down to $\sigma(K)$. Two definitions are natural

$$\underline{\mu}(A) = \inf\{L(\mu)(A'): A' \in A^\theta\}$$

$$\bar{\mu}(A) = \sup\{L(\mu)(A'): A' \in A^\theta\}.$$

It is not hard to find examples which show that $\bar{\mu}$ need not be a measure; - on the other hand we have

Proposition 2: $\underline{\mu}$ is a measure on $\sigma(K)$.

Proof: Let $\{A_n\}$ be a disjoint sequence of sets in $\sigma(K)$, and let $A = \bigcup A_n$. Choose $A' \in A^\theta$ such that $L(\mu)(A') \leq \underline{\mu}(A) + \epsilon$. Let $B_n \in A_n^\theta$, and define $A'_n = (A \cap B_n) \setminus \bigcup_{m < n} B_m$. Then the A'_n 's are disjoint and $A'_n \in A_n^\theta$. Hence

$$\underline{\mu}(A) \geq L(\mu)(A') - \epsilon \geq \sum L(\mu)(A'_n) - \epsilon \geq \sum \underline{\mu}(A_n) - \epsilon.$$

Since we can do this for arbitrary positive ϵ , $\underline{\mu}(A) \geq \sum \underline{\mu}(A_n)$.

To prove the opposite inequality, choose $A'_n \in A_n^\theta$ such that $L(\mu)(A'_n) \leq \underline{\mu}(A_n) + \frac{\epsilon}{2^{n+1}}$. Then $\bigcup A'_n \in A^\theta$, and hence

$$\underline{\mu}(A) \leq L(\mu)(\bigcup A'_n) \leq \sum L(\mu)(A'_n) \leq \sum \underline{\mu}(A_n) + \epsilon.$$

It follows that $\underline{\mu}(A) \leq \sum \underline{\mu}(A_n)$, and the proposition is proved.

Starting with an internal measure μ on X , we have now constructed a standard measure $\underline{\mu}$ on Y . However, the following example indicates that $\underline{\mu}$ may not have all the properties we want it to have.

Example 3: Let $X = {}^*Y = {}^*[-1,1]$, let K be the compact subsets of Y , and let $K' = {}^*K$. Let μ be the unit mass at the infinitesimal $\epsilon > 0$. Since $Y = [-1,0] \cup \bigcup_{n \in \mathbb{N}} [\frac{1}{n}, 1]$, we have ${}^*[-1,0] \cup \bigcup_{n \in \mathbb{N}} {}^*[\frac{1}{n}, 1] \in Y^0$, and hence $\underline{\mu} = 0$. But for most applications the natural standard counterpart of μ would be the δ -measure at 0. In the next section we shall show how such collapses of $\underline{\mu}$ may be avoided.

3. Pushing down Loeb-measures

We have seen that the measure $\underline{\mu}$ has a tendency to collapse; - in this section we shall first give conditions on the representation K' for this not to happen, and then show how representations satisfying these conditions can be constructed. The idea is to ensure that the measure is "large" on the elements in the generator set K , and that it is determined by its values on these sets.

From now on K will be a (\emptyset, \cup, \cap) -paving on Y , i.e. a family of subsets of Y containing \emptyset and closed under finite unions and countable intersections. A measure ν on $\sigma(K)$ is called K -regular if for all $A \in \sigma(K)$ and all $\alpha < \nu(A)$ there is a $K \in K$, $K \subset A$, such that $\alpha < \nu(K)$. Let K' be an internal representation of K , and μ an internal, finitely additive measure on X such that all elements in $\sigma(K')$ are $L(\mu)$ -measurable, - we wish to push $L(\mu)$ down to a K -regular measure on Y . Assume that $\lambda = \sup\{L(\mu)(K') : K \in K\}$ is finite. Since our measure shall be K -regular, it must have total mass λ ; - to get rid of the extra mass of $L(\mu)$ we proceed as follows. By polysaturation there exists an internal

μ -measurable set M , containing all elements of K' , with ${}^0\mu(M) \leq \lambda$. Define μ_M by $\mu_M(A) = \mu(A \cap M)$ for all μ -measurable sets A ; - obviously $L(\mu_M)(K') = L(\mu)(K')$ for all $K \in K$. If μ_M is the measure on $\sigma(K)$ constructed from μ_M as in the last section, we now seek conditions which make μ_M a K -regular measure.

A set $A \in \sigma(K')$ is called K' -approximated if for all $\varepsilon > 0$ there exist $K_1', K_2' \in K'$ such that $L(\mu)(K_1' \cup K_2') < \lambda - \varepsilon$ and $K_1 \subset \theta(A)$, $K_2 \subset \theta(A)$. The representation K' is called μ -consistent if $L(\mu)(\emptyset') = 0$ and for all $K_1, K_2 \in K$, $L(\mu)(K_1' \cap K_2') = L(\mu)((K_1 \cap K_2)')$ and $L(\mu)(K_1' \cup K_2') = L(\mu)((K_1 \cup K_2)')$. It is called μ - σ -consistent if in addition for all sequence $\{K_n\}$ of sets from K ,
$$L(\mu)((\bigcap_{n \in \mathbb{N}} K_n)') = \lim_{n \rightarrow \infty} L(\mu)(\bigcap_{m \leq n} K_m').$$

Lemma 4: Assume that μ_M is finite and that K' is μ - σ -consistent. Then the family of K' -approximated sets form a σ -algebra.

Proof: By definition of λ , the empty set is in the family A of K' -approximated sets, and a set is obviously in A if and only if its complement is. Let $\{A_n\}$ be a sequence from A , and for each n let $K_n, C_n \in K$ be such that $L(\mu)(K_n' \cup C_n') > \lambda - \varepsilon 2^{-(n+2)}$, $K_n \subset \theta(A_n)$, $C_n \subset \theta(A_n)$. As an outer approximation for $\bigcup A_n$, we choose $(\bigcap_{n \in \mathbb{N}} C_n)'$, and as an inner approximation for $\bigcup A_n$, we take $(\bigcup_{n \leq m} K_n)'$ for some $m \in \mathbb{N}$ such that $L(\mu)((\bigcup_{n \leq m} K_n)') \geq L(\mu)(\bigcup_{n \in \mathbb{N}} K_n') - \frac{\varepsilon}{2}$.

Obviously $\bigcap_{n \in \mathbb{N}} C_n \subset \theta(\bigcup_{n \in \mathbb{N}} A_n)$, $\bigcup_{n \leq m} K_n \subset \theta(\bigcup_{n \in \mathbb{N}} A_n)$, and by μ - σ -consistency $L(\mu)((\bigcap_{n \in \mathbb{N}} C_n)' \cup (\bigcup_{n \leq m} K_n')) > \lambda - \varepsilon$. This proves the lemma.

Lemma 5: Assume μ_M finite, K' μ -consistent, let $A \in \sigma(K)$, and let A_1', A_2' be two K' -approximated sets in A^θ . Then $L(\mu_M)(A_1') = L(\mu_M)(A_2')$.

Proof: Assume $L(\mu_M)(A_1') < L(\mu_M)(A_2')$, then there exist an inner approximation K' of A_2' and an outer approximation C' of A_1' such that $L(\mu_M)(K' \cap C') \neq 0$. But on the other hand $K \cap C = \emptyset$, and so by μ -consistency $L(\mu)(K' \cap C') = L(\mu)((K \cap C)') = L(\mu)(\emptyset') = 0$, and we have a contradiction.

Proposition 6: Assume that μ_M is finite, that K' is μ - σ -consistent, and that all elements in K' are K' -approximated. Then μ_M is a K -regular measure on $\sigma(K)$, and $\mu_M(K) = L(\mu_M)(K')$ for all $K \in K$.

Proof: By Lemma 4 all elements in $\sigma(K')$ must be K' -approximated, and by Lemma 5 $\mu_M(A) = L(\mu_M)(A')$ for all $A' \in A^\theta$. Since all elements in $\sigma(K')$ are K' -approximated, μ_M must be K -regular.

The result above gives us a method for pushing down the measure $L(\mu)$ on $\sigma(K')$ to a K -regular measure μ_M on $\sigma(K)$ in such a way that $\mu_M(K) = L(\mu)(K')$. The problem is that there is no reason to believe that the sets in K' should automatically be K' -approximated; - indeed, much of the trouble in Example 3 stems from the fact that this is not so. Let us now show how starting with one representation K' of K - which may be naturally given, e.g. by $K' = *K$ - we may under rather weak conditions produce a new representation K'' of K satisfying the assumptions of Proposition 6. What we need is the following slight strengthening of μ - σ -consistency: An internal f.a. measure μ is called σ -smooth at \emptyset with respect to K' if K'

is μ -consistent, and for all decreasing sequences $\{K_n\}$ of elements from K with $\bigcap K_n = \emptyset$ and all $\epsilon > 0$, there exists a $C_\epsilon \in K$ in the complement of some K_n , with $L(\mu_M)(CC'_\epsilon) < \epsilon$.

Theorem 7: Let K be an infinite (\emptyset, \cup, \cap) -paving on Y , and let K' be an internal representation of K on X . Let μ be an internal, f.a. measure on X such that $L(\mu_M)$ is finite and all elements in K' are Loeb-measurable. Assume that μ is σ -smooth at \emptyset with respect to K' . Then there exists an internal representation K'' on X which is μ - σ -consistent and such that all elements of K'' are K'' -approximated. Moreover, $L(\mu_M)(K'') = \inf\{L(\mu_M)(CC') : C \in K, K \cap C = \emptyset\} \geq L(\mu)(K')$.

Proof: We first use a trick to secure that K'' will really be an internal representation, i.e. that the assumption of Theorem 1 is satisfied. Since $\sigma(K)$ is infinite, we may find an infinite, internal set Y_0 of Loeb-measure zero. Let S be the set of all finite sequences $K_1, \dots, K_n, CC_1, \dots, CC_n$ with $K_i, C_j \in K$ such that $K_1 \cap \dots \cap K_n \cap CC_1 \cap \dots \cap CC_m \neq \emptyset$, and choose distinct elements $X_{K_1, \dots, K_n, CC_1, \dots, CC_n} \in Y_0$ for all elements in S . A $K \in K$ is said to occur positively in $K_1, \dots, K_n, CC_1, \dots, CC_m$ if $K = K_i$ for some i , and to occur negatively if $K = C_j$ for some j . Using polysaturation we find internal, disjoint subsets P_K and N_K of Y_0 such that all elements of Y_0 associated with positive occurrences of K are in P_K , and all elements associated with negative occurrences of K are in N_K .

Given $K \in K$, we shall now construct K'' . For all $C \in K$, let \hat{C} be a μ -measurable set such that $L(\mu_M)(\hat{C}C') = 0$. Let

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$\lambda_K = \inf\{L(\mu_M)(C \cap K') : C \cap K = \emptyset\}$. For all $C_1, \dots, C_n \in K$ with $C_1 \cap K = \emptyset$ and all $\epsilon > 0$, there exists an internal μ -measurable set $A \subset \hat{C}_1 \cap \dots \cap \hat{C}_n$ with $\mu(A) > \lambda_K - \epsilon$ (- in fact we may choose $A = \hat{C}_1 \cap \dots \cap \hat{C}_n$) and by saturation we get the existence of an internal $A \subset \cap\{\hat{C} : C \in K, C \cap K = \emptyset\}$ with $\mu(A) \approx \lambda_K$. Define $K'' = (A \cup P_K) \setminus N_K$, and put $K'' = \{K'' : K \in K\}$.

Having now defined K'' , we must show that it satisfies the theorem. That K'' is an internal representation is immediate from the construction of P_K and N_K , - and that all elements of K'' are K'' -approximated follows just as easily from the definition of K'' . That K'' is μ -consistent follows from the definition of μ_M and the μ -consistency of K' . Let finally $\{K_n\}$ be a sequence of elements from K ; let $C_n = \bigcap_{m \leq n} K_m$ and $K = \bigcap_{n \in \mathbb{N}} K_n$. Given $\epsilon > 0$, choose $C \in K$, $K \cap C = \emptyset$, such that $L(\mu_M)(C \cap K') < \lambda_K + \epsilon$. Then

$\bigcap_{n \in \mathbb{N}} (C_n \cap C) = \emptyset$, and by σ -smoothness at \emptyset , there exists a $C_\epsilon \in K$ with $L(\mu_M)(C'_\epsilon) > \lambda - \epsilon$ and $C_\epsilon \cap (C_n \cap C) = \emptyset$ for large enough $n \in \mathbb{N}$. Then $C_n \cap (C \cap C_\epsilon) = \emptyset$, and since $L(\mu_M)(C \cap C_\epsilon) \leq \lambda_K + 2\epsilon$, we get $L(\mu_M)((\bigcap_{n \in \mathbb{N}} K_n)') \geq \lim_{n \rightarrow \infty} L(\mu_M)(C'_n)$, and the theorem follows.

Notice the following slight strengthening of Theorem 7 that will be useful in Section 5: Assume that $C \subset K$ also is a (\emptyset, Uf, Nc) -paving, and assume that the condition of σ -smoothness only is satisfied when the sequence $\{K_n\}$ consists of elements from C , but that in addition for all $\epsilon > 0$ there is a $C \in C$ with $\inf_{K \in K} \{L(\mu_M)(K') : K \in K, K \cap C = \emptyset\} < \epsilon$. We see from the proof of Theorem 7 that C'' is μ - σ -consistent and that all elements of C'' are C'' -approximated.

Combining Proposition 6 and Theorem 7 we get:

Corollary 8: Let K' be an internal representation of K , and let μ be an internal, f.a. measure such that all elements in K' are $L(\mu)$ -measurable and have finite measure. Assume further that μ is σ -smooth at \emptyset with respect to K' . Then there exists a K -regular measure ν on $\sigma(K)$ such that

$$\nu(K) = \sup_{D \in K} \inf \{L(\mu)(C' \cap D') : C \in K, C \cap K = \emptyset\} \geq L(\mu)(K).$$

Proof: If $\lambda = \sup\{L(\mu)(K') : K \in K\}$ is finite, the theorem follows from Proposition 6 and Theorem 7. If $\lambda = \infty$, define $\nu(A) = \sup\{\mu_D(A) : A \in K\}$. The corollary then follows from the finite case.

Returning to Example 3 for a moment, we see that using Corollary 8 we get the "right" measure on $[-1,1]$.

However, there is more to the method presented above than is summarized in the corollary; using the σ -homomorphisms θ'' constructed from K'' by Theorem 1, we may push random variables and processes living on $\sigma(K'')$ down to $\sigma(K)$, and also lift variables and processes from $\sigma(K)$ to $\sigma(K'')$. In fact, it is easy to show that given a $\sigma(K)$ -measurable random variable f , we may find a $\sigma(K'')$ -measurable function $f^{\theta''}$ such that $\theta''\{f^{\theta''} > \alpha\} = \{f > \alpha\}$ for all $\alpha \in \mathbb{R}$; also, starting with a $\sigma(K'')$ -measurable function f , we construct a $\sigma(K)$ -measurable $f_{\theta''}$, with $\{f_{\theta''} > \alpha\} = \theta''\{f > \alpha\}$. Hence we can use nonstandard techniques to construct limit variables and processes, and then use θ'' to push the results down to our standard space. For an exploitation of these ideas, see [7] and its addendum [8].

4. An application to measure extensions

To illustrate the use of the theory developed above, we now give two applications to standard measure theory. The first one deals with extensions of regular measures.

Theorem 9: Let K be a (\emptyset, \cup, \cap) -paving on a set Y , and let A be an algebra of subsets of $\sigma(K)$. Let μ be a K -regular, f.a. measure on A , and assume that all elements of $K \cap A$ have finite measure. Assume further that for all decreasing sequences $\{K_n\}$ of elements from K with $\bigcap K_n = \emptyset$ and all $\epsilon > 0$, there is a $C \in A \cap K$ disjoint from some K_n with $\mu(C) < \epsilon$. Then μ may be extended to a K -regular measure on $\sigma(K)$.

Proof: If we can prove the theorem when μ is finite, the general result follows from this by putting $\nu = \sup\{\nu_K : K \in K \cap A\}$, where ν_K is the measure obtained by replacing μ by μ_K .

Assume μ finite. For all $K_1, \dots, K_n \in K$ there exists a hyperfinite algebra on $*Y$ containing $*K_1, \dots, *K_n$; by polysaturation it then exists a hyperfinite algebra B containing all $*K, K \in K$. Now $*\mu$ defines a $*$ -measure on $*A \cap B$, and since B is hyperfinite, this $*$ -measure is trivially extended to a $*$ -measure $\tilde{\mu}$ on B . Choosing $K' = *K$, we see that the conditions of Corollary 8 are satisfied, and we obtain a K -regular measure ν on $\sigma(K)$. Since μ is K -regular, it follows from the formula for ν in Corollary 8 that ν is an extension of μ .

The result above is an extension of the main theorem of Henry [5] (who proved it when K are the compact sets of some topology), and of Theorem 1 of Topsøe [17] (who proved it for the case when

\mathcal{A} is the algebra generated by \mathcal{K}). The hyperfinite partition trick employed in the proof goes back to Loeb [10].

5. An application to weak convergence

If \mathcal{K} is a (\emptyset, \cup, \cap) -paving on Y , let $\mathcal{G}_{\mathcal{K}} = \{G : G \in \mathcal{K}\}$, and define $\mathcal{K}_c = \{C \subseteq Y : C \cap K \in \mathcal{K} \text{ for all } K \in \mathcal{K}\}$; obviously \mathcal{K}_c is a (\emptyset, \cup, \cap) -paving. We say that \mathcal{K} is separated if for all disjoint $K_1, K_2 \in \mathcal{K}$, there are disjoint $G_1, G_2 \in \mathcal{G}_{\mathcal{K}_c}$ with $G_1 \supset K_1$, $G_2 \supset K_2$. A subclass $\hat{\mathcal{K}}$ of \mathcal{K}_c is said to dominate \mathcal{K} if for all $K \in \mathcal{K}$ there is a $\hat{K} \in \hat{\mathcal{K}}$ with $\hat{K} \cap K = \emptyset$.

A net $\{\mu_\alpha\}_{\alpha \in A}$ of finite, \mathcal{K} -regular measures on $\sigma(\mathcal{K})$ is said to be \mathcal{K} -weakly convergent if there is a finite, \mathcal{K} -regular measure μ on $\sigma(\mathcal{K})$ such that $\lim_\alpha \mu_\alpha(Y) = \mu(Y)$ and $\lim_\alpha \mu_\alpha(K) \leq \mu(K)$ for all $K \in \mathcal{K}$ (and hence for all $K \in \mathcal{K}_c$). The net is said to be \mathcal{K} -weakly compact if all subnets have convergent subnets. We shall prove the following characterization:

Theorem 10: Let \mathcal{K} be a separated (\emptyset, \cup, \cap) -paving on Y . A net $\{\mu_\alpha\}_{\alpha \in A}$ is \mathcal{K} -weakly compact if and only if the following conditions hold.

- (i) $\overline{\lim}_\alpha \mu_\alpha(Y) < \infty$
- (ii) For all subclasses $\hat{\mathcal{K}}$ of \mathcal{K}_c dominating \mathcal{K} and all $\varepsilon > 0$, there exists a finite subset $\hat{K}_1, \dots, \hat{K}_n$ of $\hat{\mathcal{K}}$ such that

$$\overline{\lim}_\alpha \min_{1 \leq i \leq n} \mu_\alpha(\hat{K}_i) \leq \varepsilon.$$

- (iii) For all decreasing sequences $\{K_n\}$ of sets from \mathcal{K} with $\bigcap_n K_n = \emptyset$ and all $\varepsilon > 0$, there exist an $n \in \mathbb{N}$ and a finite subset

G_1, \dots, G_m of G_{K_C} such that $K_n \subset \cap G_i$ and

$$\overline{\lim}_\alpha \min_i \mu_\alpha(G_i) \leq \varepsilon.$$

Proof: Assume that (i) - (iii) hold, and that $\{v_\beta\}_{\beta \in B}$ is a subnet of $\{\mu_\alpha\}_{\alpha \in A}$. Let ω be an infinite element in *B , i.e. an element larger than all elements of B , and let \tilde{v}_ω be the corresponding internal measure in ${}^*(\{v_\beta\}_{\beta \in B})$. Let $\lambda = {}^0\tilde{v}_\omega(*Y)$, by (i) λ is finite. Applying (iii) with all $K_n = \emptyset$, we see that for all $\varepsilon > 0$ there is a $\hat{K} \in K_C$ with $\tilde{v}_\omega(*\hat{K}) > \lambda - \varepsilon$. By (ii), we also see that for all positive ε , there exist a $K \in K$ such that $\inf_{\hat{K} \in K_C, \hat{K} \cap K = \emptyset} \{L(\tilde{v}_\omega)(* \hat{K})\} < \varepsilon$. Combining this with (iii), we see that we may apply the remark following the proof of Theorem 7 to obtain a representation of K satisfying Proposition 6. Hence we get a K -regular measure v on $\sigma(K)$ with $v(Y) = L(\tilde{v}_\omega)(*Y)$ and $v(K) \geq L(\tilde{v}_\omega)(K)$ for all $K \in K$, and, using the transfer principle, it is not hard to construct a subnet of $\{v_\beta\}$ converging to v . This proves the sufficiency of (i) - (iii).

For completeness we also include a proof of the necessity, although it contains no nonstandard ideas. Assume that $\{\mu_\alpha\}$ is compact; then (i) is obviously satisfied. If (iii) is not satisfied, let $\{K_n\}$ and $\varepsilon > 0$ be a sequence and a number that violate it. We construct a subnet $\{v_\beta\}_{\beta \in B}$ of $\{\mu_\alpha\}_{\alpha \in A}$ as follows: Let B be the set of all triples $\langle \alpha, n, \{G_1, \dots, G_p\} \rangle$ with $\alpha \in A$, $n \in \mathbb{N}$, $K_n \subset \cap G_i$, and $\min_i \mu_\alpha(G_i) > \varepsilon$. Order B to a directed set by letting

$$\langle \alpha, n, \{G_1, \dots, G_p\} \rangle \leq \langle \alpha', n', \{G'_1, \dots, G'_{p'}\} \rangle$$

if $\alpha \leq \alpha'$, $n \leq n'$, and $\{G_1, \dots, G_p\} \subseteq \{G'_1, \dots, G'_{p'}\}$. Let

$\nu_{\langle \alpha, n, \{G_1, \dots, G_p\} \rangle} = \mu_\alpha$; then $\{\nu_\beta\}_{\beta \in B}$ is a subnet of $\{\mu_\alpha\}$.

Assume now that $\{\nu_{\beta_\gamma}\}_{\gamma \in \Gamma}$ is a subnet of $\{\nu_\beta\}$ converging to ν . Since $\cap K_n = \emptyset$, there exists an $n \in \mathbb{N}$ such that $\nu(K_n) < \frac{\varepsilon}{2}$; since ν is K -regular, there is a $G \in \mathcal{G}_K$, $G \supset K_n$, such that $\nu(G) < \frac{\varepsilon}{2}$; - and since K is separated, there exist $K' \in K_c$, $G' \in \mathcal{G}_{K_c}$, such that $K_n \subset G' \subset K' \subset G$. By construction of $\{\nu_{\beta_\gamma}\}$, $\nu_{\beta_\gamma}(K') \geq \nu_{\beta_\gamma}(G') > \varepsilon$ for all large enough γ . Since $\nu(K') \geq \lim_{\gamma} \nu_{\beta_\gamma}(K') \geq \varepsilon$, it follows that $\nu(G) \geq \varepsilon$, and we have a contradiction. Hence (iii) is necessary for $\{\mu_\alpha\}$ to be compact.

Assume finally that (ii) is not satisfied. Letting \hat{K}_c and $\varepsilon > 0$ be a class and a number that violate it, we may construct a subnet $\{\nu_\beta\}_{\beta \in B}$ of $\{\mu_\alpha\}$ such that for all $K \in K$ there is a $\hat{K} \in \hat{K}_c$, $\hat{K} \cap K = \emptyset$, such that $\nu_\beta(\hat{K}) > \varepsilon$ for large enough β . Let $\{\nu_{\beta_\gamma}\}_{\gamma \in \Gamma}$ be a subnet of $\{\nu_\beta\}$ converging to ν . Then

$$\sup_{K \in K} \nu(Y \setminus K) \geq \sup_{K \in K} \nu(\hat{K}) \geq \sup_{K \in K} \lim_{\gamma} \nu_{\beta_\gamma}(\hat{K}) \geq \varepsilon,$$

which is impossible since ν should be K -regular. This proves the theorem.

The result above may be considered as a generalization of the famous theorem by Prohorov [14], characterizing the weakly compact sets of measures on complete, separable metric spaces. It is an extension of Theorem 4 of Topsøe [17]. Nonstandard treatments of Prohorov-type theorems on topological spaces have been given by Anderson and Rashid [3], Loeb [12], and Lindstrøm [9].

6. Comparison with the standard part construction

For the case where $X = {}^*Y$ and Y is a topological space, we now compare the method of Section 3 with the usual method for pushing down Loeb-measures using the standard part map.

Proposition 11: Let (Y, τ) be a Hausdorff space, and let K be the paving of compact sets. Let μ be a finite, f.a. internal measure on Y such that *G is μ -measurable for all $G \in \tau$. Then $st^{-1}(K)$ is $L(\mu_M)$ -measurable for all $K \in K$, and $v_{st}(A) = L(\mu_M)(st^{-1}(A))$ is a K -regular measure on $\sigma(K)$. Moreover, putting $K' = {}^*K$ for all $K \in K$, the conditions of Corollary 8 are satisfied, and v_{st} equals the v of that result.

Proof: For $K \in K$ we have

$$st^{-1}(K) = \cap \{ {}^*G : G \in \tau \text{ and } K \subset G \}.$$

If $\lambda_K = \inf \{ {}^0\mu_M({}^*G) : G \in \tau \text{ and } K \subset G \}$, we see by polysaturation that for all $\epsilon > 0$, there is a μ -measurable $G_\epsilon \in {}^*\tau$, $G_\epsilon \subset st^{-1}(K)$, with $\mu_M(G_\epsilon) > \lambda_K - \epsilon$. Hence $st^{-1}(K)$ is $L(\mu_M)$ -measurable with $L(\mu_M)(st^{-1}(K)) = \lambda_K$. By definition of μ_M there are elements in K with * -versions of arbitrary large ${}^0\mu_M$ -measure, and hence $\lambda_K = \inf \{ {}^0\mu_M({}^*G) : K \subset G \text{ and } (G \in K) \}$. But this is exactly the formula for v in Corollary 8, so if we can only show that v_{st} is regular, and that the conditions of Corollary 8 are satisfied, the proposition will follow.

If $A \in \sigma(K)$, there is an internal, nearstandard, μ -measurable $B \subset A$ with $L(\mu_M)(B) > L(\mu_M)(st^{-1}(A)) - \epsilon$. Then $st(B) \subset A$ is compact, and $st^{-1}(A) \supset st^{-1}st(B) \supset B$. Hence $v_{st}(st(B)) \geq v_{st}(A) - \epsilon$, and v_{st} is regular.

Since the conditions of Corollary 8 is trivially satisfied by the finite intersection property of the compact sets, the proposition follows.

Although this result shows that our method is more general than the standard part method, it is convenient to use that method whenever possible, since the standard part map is much easier to work with than the abstract θ discussed at the end of Section 3. The standard part arguments in the proof go back to Anderson [2].

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